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THE INSTABILITY OF STEADY FLOWS GENERATED BY A VORTEX LINE IN A STRATIFIED GAS†

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The internal waves that arise in an ideal uniformly stratified gas due to a vertical vortex line are investigated. In the case of slow motion, when acoustic oscillations can be neglected compared with buoyancy oscillations, the problem reduces to solving a mixed problem for a non-linear third-order partial differential equation. A steady solution is found and in a linear approximation it is proved to be unstable. © 1999 Elsevier Science Ltd. All rights reserved.

1. THE BASIC DIFFERENTIAL EQUATION

Consider a uniformly stratified ideal gas occupying three-dimensional space, with the z axis of a cylindrical system of coordinates r, θ, z in the opposite direction to the gravity force, and consider the gas motion which is symmetric about the z axis. If the motion is sufficiently slow, i.e. the change of the velocity of sound in a fluid particle a is small compared with a, the equation of continuity has the same form as for an incompressible fluid. In an equilibrium position, the entropy is constant in horizontal planes and increases with height. For a fluid particle which at a time t is a point with coordinates (r, θ, z), the distance $\zeta(r, t, z)$ of that particle from the equilibrium position to a fixed horizontal plane is uniquely defined. In the particle, the function $\zeta(r, t, z)$ is conserved. The quantity $w = z - \zeta(r, t, z)$ gives the deviation along the vertical of the fluid particle from the equilibrium position.

We will consider the class of the fluid motions for which the radial velocity $v_r = 0$, while the vertical deviation from the equilibrium position and the velocity of sound depend only on r and t. Since entropy in the particle is conserved, $p = \alpha(\zeta(r, t, z))\rho^{\varkappa}$, where p is the pressure, ρ is the density and $\varkappa = c_p/c_v$, the function $\alpha(\zeta)$ gives the entropy distribution at the equilibrium position. The square of the Väisälä–Brunt frequency $N^2 = g\alpha'(\zeta)/\varkappa\alpha(\zeta)$ in a uniformly stratified gas has a constant value, where g is the acceleration due to gravity.

With the above assumptions the equations of motion take the form [1]

$$\frac{v_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}, \quad \frac{\partial^2 w}{\partial t^2} = -g\frac{1}{\rho}\frac{\partial p}{\partial z}, \quad \frac{\partial (rv_\theta)}{\partial t} = 0 \tag{1.1}$$

$$p = \alpha(z - w(r,t))\rho^{\varkappa}$$

We transform system of equations (1.1). In this case the argument used in [2] is simplified. It follows from the third equation that rv_θ is an arbitrary function of r. In particular, if $rv_\theta = \Gamma/(2\pi)$ we obtain a vertical vortex line. If we use the fourth equation, the first two equations take the form

$$\frac{v_\theta^2}{r} = \frac{AN^2}{g}\frac{\partial w}{\partial r} + \frac{\partial A}{\partial r}, \quad \frac{\partial^2 w}{\partial t^2} + g = \frac{N^2 A}{g}, \quad A = \frac{a^2}{\varkappa - 1} \tag{1.2}$$

Eliminating the function A from system (1.2) and making the replacement of variables

$$w = \frac{g}{N^2}W, \quad Nt = \tau, \quad u = -\frac{N^2}{g^2}\int_{+\infty}^{r}\frac{v_\theta^2(x)}{x}dx \tag{1.3}$$

we refuse (1.2) to the form

$$\frac{\partial}{\partial u}\left(\frac{\partial^2 W}{\partial \tau^2} + W + u\right) + \frac{\partial W}{\partial u}\frac{\partial^2 W}{\partial \tau^2} = 0 \tag{1.4}$$

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The integral in (1.3) is assumed to converge. Note that Eq. (1.4) is independent of the form of the function υ_θ. Equation (1.4) has the steady solution $W_0 = -u$.

2. INVESTIGATION OF THE LINEAR INSTABILITY OF THE STEADY SOLUTION

Putting

$$W = -u + e^u \omega(u,\tau)$$

in Eq. (1.4) and retaining only linear terms in ω, we obtain a linear equation for the perturbations

$$\frac{\partial}{\partial u}\left(\frac{\partial^2 \omega}{\partial \tau^2} + \omega\right) + \omega = 0 \tag{2.1}$$

At the initial time $\tau = 0$, the vertical deviations and vertical velocities are specified. Since the value $u = 0$ corresponds to $r = \infty$, and there are no perturbations at infinity, the function ω vanishes when $u = 0$. We have

$$\omega(u,0) = \omega_0(u), \quad \frac{\partial \omega}{\partial \tau}(u,0) = \omega_1(u), \quad \omega(0,\tau) = 0 \tag{2.2}$$

where $\omega_0(u)$ and $\omega_1(u)$ are specified functions.

Applying a Laplace transformation with respect to time to the solution of problem (2.1), (2.2), we obtain a Cauchy problem for a first-order ordinary differential equation for the Laplace transform $\omega^*(u,p)$, and it has a solution of the form

$$\omega^* = \frac{1}{1+p^2}\int_0^u (p\omega_0(v) + \omega_1(v))\exp\left(-\frac{u-v}{1+p^2}\right)dv$$

Applying the inverse transformation and using the theory of residues, we obtain

$$\begin{aligned} \omega(u,\tau) &= -\int_0^u \left(\frac{\partial^2 G(u-v,\tau)}{\partial u \partial t}\omega_0(v) + \frac{\partial G(u-v,\tau)}{\partial u}\omega_1(v)\right)dv \\ G(u,\tau) &= -\operatorname{Im}\left(\frac{1}{\pi}\int_{C(i)} \exp\left(p\tau - \frac{u}{1+p^2}\right)dp\right) \end{aligned} \tag{2.3}$$

where $C(i)$ is a circle of radius less than unity with centre at the point i.

It follows from formula (2.3) that the behaviour of the function $\omega(u,\tau)$ is governed by the behaviour of Green's function $G(u,\tau)$. Expanding the function $1/(1+p^2)$ in elementary fractions, making the replacement of the variable of integration $\zeta = p - i$ and putting

$$\beta = \sqrt{2u\tau}e^{-i\pi/4}, \quad \alpha = \sqrt{\frac{u}{2\tau}}e^{-i\pi/4} \tag{2.4}$$

we can represent the function $G(u,\tau)$ in the form

$$G(u,\tau) = -\frac{1}{\pi}\operatorname{Im}\alpha e^{i\tau - u/4}\left(\int_{C(0)} \Phi_1(\zeta,\beta)\Phi_2(\zeta,\alpha)d\zeta\right)$$

$$\Phi_1(\zeta,\beta) = \exp\left(\frac{\beta}{2}\left(\zeta - \frac{1}{\zeta}\right)\right), \quad \Phi_2(\zeta,\alpha) = \exp\left(\frac{u\alpha\zeta/(2i)}{4(1+\alpha\zeta/(2i))}\right)$$

Using generating functions for the Laguerre polynomials and Bessel functions [2] we obtain

$$G(u,\tau) = \operatorname{Im}(2ie^{i\tau - u/4}\alpha J_1(\beta)) + \operatorname{Im}\left(e^{i\tau - u/4}\sum_{n=2}^{\infty}\frac{\alpha^n}{(2i)^{n-2}}J_n(\beta)\left(L_{n-1}\left(\frac{u}{4}\right) - L_{n-2}\left(\frac{u}{4}\right)\right)\right)$$

Green's function can be written in real form using the Kelvin functions $\mathrm{ber}_n(x)$ and $\mathrm{bei}_n(x)$. If the time t is fixed and $u \to 0$, from the fact that $J_1(\beta) \approx \beta/2$ as $\beta \to 0$, we find that $G(u, \tau) \approx \sin\tau$.

If the parameter $u\tau \to \infty$, then, using the asymptotic formula for $J_1(z)$ we obtain

$$J_1(\sqrt{2u\tau e}^{-i\pi/4}) \approx \frac{1}{\sqrt{2\pi i}(2u\tau)^{1/4}} \exp\left(\sqrt{u\tau} - i\frac{3\pi}{8} + i\sqrt{u\tau}\right)$$

Substituting this expression into the formula for $G(u, \tau)$, we obtain the principal term of the asymptotic representation in the form

$$G(u,\tau) \approx \frac{1}{\sqrt{\pi}(u\tau)^{3/4}} \exp\left(\sqrt{u\tau} - \frac{u}{4}\right) \cos\left(\tau + \sqrt{u\tau} - \frac{5\pi}{8}\right)$$

This formula shows that the amplitude of the oscillations to infinity as $u\tau \to \infty$, which indicates that the equilibrium state of a fluid with a stationary vortex line is unstable. For small values of the Väisälä–Brunt frequency and at large distances from the vortex line, the instability develops slowly, and only at long times are the simple harmonic oscillations replaced by oscillations with increased amplitude. It would be interesting to investigate the stability of equilibrium in a non-linear formulation.

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